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AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZN 69/76 DECEMBER

J. DE VRIES

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amsterdam 1976

stichting
mathematisch
centrum

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2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) : 54H15, 54E15, 54A25, 54D35

Bounds for a cardinal function on G-spaces

by

J. de Vries

ABSTRACT

Let G be a locally compact topological group. For every Tychonoff G-space $\langle X,\pi\rangle$ we define $b\langle X,\pi\rangle$ as the least cardinal number of a base of a uniformity for X with respect to which π is motion-equicontinuous. We show in this note that $\ell w(G) \leq b\langle X,\pi\rangle \leq w(X)$, where ℓw and w denote the local weight and the weight function, respectively.

KEY WORDS & PHRASES: *G-space, locally compact topological group, motion-equicontinuity, boundedness for G-spaces, weight, local weight, uniform weight*

1. INTRODUCTION

In this note the letter G will always denote a locally compact topological group with unit e. Recall that a G-*space* is an ordered pair $\langle X,\pi\rangle$, where X is a topological space and $\pi: G \times X \to X$ is a continuous mapping such that $\pi(e,x) = x$ and $\pi(t,\pi(s,x)) = \pi(ts,x)$ for all $t,s \in G$ and $x \in X$. We shall use the following notation: $\pi^t x := \pi(t,x) =: \pi_x t$ for $(t,x) \in G \times X$. The G-space $\langle X,\pi\rangle$ is called *effective* whenever $\pi^t \neq \pi^e$ for $t \neq e$. In the sequel we shall use only Tychonoff G-spaces, i.e. G-spaces $\langle X,\pi\rangle$ where X is a Tychonoff (= completely regular Hausdorff) space. If $\mathcal{U}$ is an admissible uniformity for X then $\langle X,\pi\rangle$ is called $\mathcal{U}$-*bounded*)1 whenever the subset $\{\pi_x : x \in X\}$ of C(G,X) is equicontinuous at e (with respect to the uniformity $\mathcal{U}$ in X, of course). In [2], Proposition 7.3.12 it has been shown that this concept of boundedness is closely related to the possible existence of a G-compactification of $\langle X,\pi\rangle$, that is, an equivariant embedding of $\langle X,\pi\rangle$ in a compact Hausdorff G-space. According to the main result in [3], there exists always a uniformity $\mathcal{U}$ for X such that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded, provided G is locally compact. In that case, the least cardinal number of a base for a uniformity $\mathcal{U}$ of X such that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded will be denoted $b\langle X,\pi\rangle$. We shall derive bounds for $b\langle X,\pi\rangle$ in terms of the local weight $\ell w(G)$ of G, the weight $w(X)$ and the uniform weight $u(X)$ of X. In addition, we touch the question whether there is any relationship between the existence of a metrizable G-compactification of $\langle X,\pi\rangle$ (in the case that X is separable and metrizable) and the value of $b\langle X,\pi\rangle$.

2. RESULTS

<u>PROPOSITION</u>. *Let* G *be a locally compact topological group. Then for every Tychonoff* G-*space* $\langle X,\pi\rangle$ *the following inequalities hold:*

$$\max\{\ell w(G), u(X)\} \leq b\langle X,\pi\rangle \leq w(X).$$

)1 Also called *motion-equicontinuous* by some authors.

PROOF. It is obvious that $u(X) \leq b\langle X,\pi\rangle$, so it is sufficient to prove that $\ell w(G) \leq b\langle X,\pi\rangle \leq w(X)$. First, we show that $\ell w(G) \leq b\langle X,\pi\rangle$ provided $\langle X,\pi\rangle$ is effective. To this end, consider an admissible uniformity $\mathcal{U}$ for X such that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded, and which has a base $\mathcal{B}$ such that $|\mathcal{B}| = b\langle X,\pi\rangle$. Define, for every $x \in X$ and $\alpha \in \mathcal{B}$,

$$V_{x,\alpha} := \{t \in G : (x,\pi_x t) \in \alpha\}.$$

Since the mapping $t \mapsto (x,\pi_x t) : G \to X \times X$ is continuous and each $\alpha \in \mathcal{B}$ is a neighbourhood of the diagonal in $X \times X$, each $V_{x,\alpha}$ is a neighbourhood of of e in G. Setting $V_\alpha := \bigcap\{V_{x,\alpha} : x \in X\}$, the fact that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded implies that V_α is a neighbourhood of e in G for every $\alpha \in \mathcal{B}$. Moreover, $\bigcap\{V_\alpha : \alpha \in \mathcal{B}\} = \{e\}$ because $\langle X,\pi\rangle$ is effective. It follows, that G is a Hausdorff group. However, G is locally compact, and now the fact that $\bigcap\{V_\alpha : \alpha \in \mathcal{B}\} = \{e\}$ implies that $\{V_\alpha : \alpha \in \mathcal{B}\}$ is a local subbase at e. Therefore, $\ell w(G) \leq |\mathcal{B}| = b\langle X,\pi\rangle$.

Next, we show that $b\langle X,\pi\rangle \leq w(X)$. Remember from the first part of the proof that G is Hausdorff. Since G is also locally compact, and G acts effectively on X, it follows that $\ell w(G) \leq w(X)$; see [4]. In [3], we constructed a uniformity $\mathcal{U}$ for X such that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded. This uniformity was generated by a set $\{g_j : j \in J\}$ of continuous, $[0,1]$-valued functions, whence $b\langle X,\pi\rangle \leq |J|$. In the construction, the index set J was, in fact, the set $\mathcal{B}_e \times C(X,[0,1])$, where $\mathcal{B}_e$ is a local base at e in G. So we may assume that $|\mathcal{B}_e| = \ell w(G) \leq w(X)$. However, the construction in [3] works equally well if we replace $C(X,[0,1])$ by any of its subsets which separates points and closed subsets of X. Since X can topologically be embedded in a product of $w(X)$ copies of $[0,1]$, there exists such a subset of $C(X,[0,1])$ of cardinality $w(X)$. Thus we may assume that $|J| \leq w(X)$, whence $b\langle X,\pi\rangle \leq w(X)$. □

REMARKS. Let $\langle X,\pi\rangle$ be a G-space.

1. If $\mathcal{U}$ is an admissible uniformity for X and if $\mathcal{B}$ is a base for $\mathcal{U}$, then we can define, for every $x \in X$ and every $\alpha \in \mathcal{B}$, as in the above proof

$$V_{x,\alpha} := \{t \in G : (x,\pi_x t) \in \alpha\};$$

$$V_\alpha := \bigcap_{x \in X} V_{x,\alpha}.$$

Obviously, $\langle X,\pi\rangle$ *is* $\mathcal{U}$*-bounded iff* V_α *is a neighbourhood of* e *in* G *for every* $\alpha \in \mathcal{B}$.

The following is easy to prove: if $\alpha \in \mathcal{B}$ and α is closed in $X \times X$, and if A is a dense subset of X, then

$$V_\alpha = \bigcap_{x \in A} V_{x,\alpha}.$$

If G is non-discrete, and the cardinal number $p(G)$ is defined as the least cardinal number of a collection of neighbourhoods of e in G whose intersection is *not* a neighbourhood of e, then the following statement is clear: *if* $d(X) < p(G)$, *then* $\langle X,\pi\rangle$ *is* $\mathcal{U}$*-bounded for every admissible uniformity* $\mathcal{U}$ *of* X (here $d(X)$ denotes the density of X). This generalizes the trivial observation that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded for every admissible uniformity if G is discrete.

2. In the second part of the proof of our proposition, i.e. the proof that $b\langle X,\pi\rangle \le w(X)$, we used the fact that G was Hausdorff (shown in the first part of the proof) and that $\langle X,\pi\rangle$ was effective. Both assumptions can be removed. Indeed, if G is locally compact (but possibly not Hausdorff) and $\langle X,\pi\rangle$ is not effective, then $H := \{t \in G : \pi^t = \pi^e\}$ is a closed, normal subgroup of G. Hence G/H is a locally compact Hausdorff group. Moreover, G/H acts effectively on X by $\sigma(tH,x) := \pi(t,x)$ $(t\in G,\ x\in X)$. So we have an effective G/H-space $\langle X,\sigma\rangle$. It is easy to see that for every admissible uniformity $\mathcal{U}$ of X the G-space $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded iff the G/H-space $\langle X,\sigma\rangle$ is $\mathcal{U}$-bounded, so that $b\langle X,\pi\rangle = b\langle X,\sigma\rangle$. But our proposition applies to the G/H-space $\langle X,\sigma\rangle$ to the effect that $b\langle X,\sigma\rangle \le w(X)$. Hence $b\langle X,\pi\rangle \le w(X)$.

3. In a similar way one shows, that if $\langle X,\pi\rangle$ is not effective, and G possibly not Hausdorff (but still locally compact), then $\ell w(G/H) \le b\langle X,\pi\rangle$.

4. In [2], 7.3.2, we defined a G-space $\langle X,\pi\rangle$ to be *metrically bounded* if X is metrizable and $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded for some metric uniformity $\mathcal{U}$ (equivalently: a uniformity with a countable base). So a G-space $\langle X,\pi\rangle$ is metrically bounded iff $b\langle X,\pi\rangle \le \aleph_0$. It was shown that if G is locally compact and sigma-compact, then $\langle X,\pi\rangle$ is metrically bounded if X is separable and metrizable. Using Remark 2 above, it is clear that sigma-compactness of G can be removed from the hypothesis: *if* G *is locally compact then every separable metrizable* G*-space* $\langle X,\pi\rangle$ *is metrically bounded.*

5. In a sense, the bounds for $b\langle X,\pi\rangle$ given in our proposition are best possible. Indeed, if G is discrete and X is metrizable but not separable, then $b\langle X,\pi\rangle = u(X) = \aleph_0$ and $b\langle X,\pi\rangle < w(X)$. On the other hand, in [2], 7.3.5 (iii) is an example of a locally compact (even sigma-compact) group G and a non-separable metrizable space X for which $b\langle X,\pi\rangle = w(X)$. Finally, if we consider a suitable locally compact group G acting on itself by left translations, we obtain a G-space $\langle G,\rho\rangle$ with $b\langle G,\rho\rangle = \ell w(G) < w(G)$ (start with a group G for which $\ell w(G) < w(G)$, and observe that (G,ρ) is $\mathcal{U}$-bounded for the right uniformity $\mathcal{U}$ of G; hence $b\langle G,\rho\rangle \leq \ell w(G)$).

3. RELATION OF $b\langle X,\pi\rangle$ TO THE SIZE OF G-COMPACTIFICATIONS

Recall from [3] that a G-*compactification* of $\langle X,\pi\rangle$ is an equivariant dense embedding of $\langle X,\pi\rangle$ in a compact Hausdorff G-space $\langle Y,\sigma\rangle$. If $\langle Y,\sigma\rangle$ is a G-compactification of $\langle X,\pi\rangle$, then clearly $b\langle X,\pi\rangle \leq u(Y)$. Indeed, since Y is compact, a straightforward compactness argument shows that $\langle Y,\sigma\rangle$ is bounded with respect to its unique uniformity $\mathcal{U}$. Then $\langle X,\pi\rangle$ is, of course, bounded with respect to the relativation of $\mathcal{U}$ to X, and $b\langle X,\pi\rangle \leq u(Y)$. However, for the compact space Y, we have $u(Y) = w(Y)$, hence $b\langle X,\pi\rangle \leq w(Y)$.
In [3], the existence of a G-compactification $\langle Y,\sigma\rangle$ of $\langle X,\pi\rangle$ has been shown such that $w(Y) \leq \max\{w(G),w(X)\}$, under the assumptions that G is locally compact and X is a Tychonoff space. Obviously, this is consistent with our proposition, but it gives no additional information about the value of $b\langle X,\pi\rangle$.

So we ask the question the other way round: can the weight of a possible G-compactification be estimated in terms of $b\langle X,\pi\rangle$? In particular, has $\langle X,\pi\rangle$ a metrizable G-compactification if $b\langle X,\pi\rangle = \aleph_0$?
The following example (which is esentially due to the late professor J. DE GROOT [1]) answers the second question in the negative, thus leaving completely open the first one.

<ins>EXAMPLE</ins>. Let X be the space of the rationals with its usual topology, and let G be the group of all homeomorphisms of X onto itself, provided with the discrete topology, the action of G on X being the obvious one.

Then $b\langle X,\pi\rangle = u(X) = \aleph_0$. We shall show that no G-compactification of $\langle X,\pi\rangle$ can be metrizable.

Let Y be an arbitrary metrizable compactification of X. Then the metric of Y induces a metric in X, and if all members of G where extendable to Y, they would be all uniformly continuous with respect to this metric. This, however, is not true: there exists a Cauchy sequence $\{x_n\}_n$ in X with respect to this metric which does not converge (X is not topologically complete). If $\{a_n\}_n$ and $\{b_n\}_n$ are sequences converging to 0 and 1 respectively, then there exists $h \in G$ such that $h(x_n) = a_n$ if n is odd and $h(x_n) = b_n$ if n is even. Then h is not uniformly continuous.

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